

Stress Propagation in Sand

M. E. Cates and J. P. Wittmer

*Department of Physics and Astronomy, University of Edinburgh
King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, UK*

PACS numbers 46.10.+z, 46.30.-i, 81.35.+k, 83.70 Fn

We describe a new continuum approach to the modelling of stress propagation in static granular media, focussing on the conical sandpile created from a point source. We argue that the stress continuity equations should be closed by means of scale-free, local constitutive relations between different components of the stress tensor, encoding the construction history of the pile: this history determines the organization of the grains, and thereby the local relationship between stresses. Our preferred model FPA (Fixed Principle Axes) assumes that the eigendirections (but not the eigenvalues) of the stress tensor are determined forever when a material element is first buried. Stresses propagate along a nested set of archlike structures within the medium; the results are in good quantitative agreement with published experimental data. The FPA model is one of a larger class, called OSL (Oriented Stress Linearity) models, in which the direction of the characteristics for stress propagation are fixed at burial. We speculate on the connection between these characteristics and the *stress paths* observed microscopically.

1 Introduction

A sandpile is normally constructed by pouring sand from a stationary point source. Each element of sand arrives at the apex of the pile, rolls down the slopes, comes to rest, and is finally buried. Thus successive layers at the angle of repose ϕ are added to the symmetrical cone as the height H of the pile increases. Some of the simplest questions one can ask about this system concern the distribution of stresses in the pile. Intuitively one would guess that the maximum vertical force would be recorded directly beneath the apex of the

★

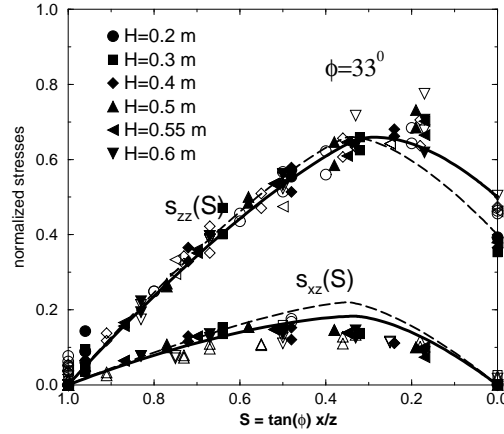


Fig. 1. Rescaled data of Smid and Novosad[1] from conical piles of different heights ($H = 20 - 60\text{cm}$) for quartz sand (closed symbols) and NPK-1 fertilizer (open symbols). Radial stress field (RSF) scaling is obeyed to experimental accuracy. The data are compared with two versions of the FPA model in three dimensions: the bold line assumes an *uniaxial* stress tensor, the dashed one a possible alternative discussed in the text.

pile; but in fact, the experiments show a pronounced *dip* in the force beneath the apex.[1,2]

In fig. 1 we show vertical normal and shear stresses measured by Smid and Novosad [1] on the supporting surface for piles of varying height for two different cohesionless Coulomb materials. We normalized the stresses by the total weight of the piles; notice the good “radial stress field” (RSF) scaling of curves from piles of different heights. These data show that to experimental accuracy, there is *no intrinsic length scale* in the sandpile problem! This first observation is one of the cornerstones in our modelling approach to the stress distribution in aggregates consisting of cohesionless hard particles held up by static frictional forces. [10,11] The stresses were measured on a linear array of sensors across the width of the piles; there are two data points for each symbol corresponding to two opposite points on the pile. Their difference gives an indication of the size of the fluctuations: Stochastic processes occurring in the propagation of stresses are apparently not so broad as to prevent one formulating the “continuum limit” for piles of reasonable size. This second observation strongly suggests a traditional continuum approach, [3,4], although there is certainly some noise. [7–9,14]

2 Systematic Continuum Approach

Continuum limit: We suppose that after sufficient coarse-graining the force

propagation can be described by a traditional “continuum limit” based on the continuity equation of the stress tensor:

$$\nabla_i \sigma_{ij} = g_j \quad (1)$$

where the source term g_j is the gravitational field (we take units where the density is unity). This equation is incomplete: in d dimensions $d(d-1)/2$ *constitutive relations* (CR) are required. These should be based on physically motivated assumptions. For simplicity we will mostly stick to the $d = 2$ dimensional case which requires one CR.

Coulomb boundary conditions: Let us first fix the boundary conditions on free surfaces. Because of the symmetry of the piles we can restrict our attention to the left side (fig. 2). The Coulomb yield criterion (that shear forces do not exceed the maximum permitted by static friction) can be written in terms of the normalized ellipticity function

$$\Upsilon \stackrel{\text{def}}{=} \frac{1}{\sin(\phi)} \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} \quad (2)$$

where ϕ is a material parameter and σ_1, σ_2 are major and minor principal stresses. In fact, it is easily established [11] that ϕ is the angle of repose for a freestanding pile whose surface is a slip plane (such as a pile created from a point source). This IFS (incipient failure at surface) boundary condition requires (a) that the yield function $\Upsilon = 1$ on the surface; and (b) that the direction of the major principal axes bisects the vertical and the surface: $\Psi = (90^\circ - \phi)/2$ where the notation is shown in fig. 2. Note that the Coulomb criterion does not (without further assumptions) fix the stresses in the bulk; here we know only that $\Upsilon \leq 1$.

No intrinsic length: In fig. 1 we have shown strong evidence that the stresses scale with height. We assume therefore that there exists no intrinsic length scale. This severely limits the possible form the missing constitutive relation can take. In particular it rules out all elastic models invoking a relation between stress and strain: associated with a stress scale (Young modulus) there is always (under gravity) a length scale. Effectively we require that this length scale is much larger than the size H of the pile and can be sent to infinity. Quantities like the direction of the tensor and its ellipticity have to scale like $\Psi(x, z) = \Psi(S)$, $\Upsilon(x, z) = \Upsilon(S)$ where $S = \tan(\phi)x/z$ is a scaling variable (defined to be unity on the free surface). This is called RSF (radial stress field) scaling.

Locality: Now we seek a CR, which (i) is scale-free (no intrinsic length or stress scale), (ii) may depend on the local packing of the grains fixed by the aggregation history, and (iii) is simple. We therefore assume the *locality* of the CR: the

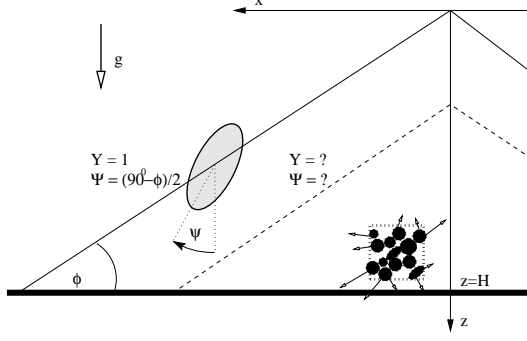


Fig. 2. A two dimensional symmetrical PS-pile of Height H . The (x, z) coordinates used are shown: The z -axis points from the apex down in the direction of gravitation. The cohesionless Coulomb material is assumed to be at IFS fixing two invariant quantities near the surface: the ellipticity of the stress tensor $\Upsilon = 1$ and its direction Ψ . The major principal axis then bisects the surface and the vertical. Nothing is known *a priori* however on both quantities on former surfaces (broken line). The invariant quantity memorized by the texture will provide the missing CR.

relation between stresses is unaffected by distant loads, although the stresses themselves do of course depend on such loads. For this to hold, we should check *a posteriori* that no yield occurs. In two dimensions this assumption, together with the earlier hypotheses requires that there is a function C , such that

$$C(\Upsilon, \Psi, S) = 0. \quad (3)$$

Perfect memory: For a pile made from a point source in two dimensions, it is intuitively clear that particles on the left of the apex may transmit stress anisotropically due to packing features *memorized* from the deposition, with particles on the right the mirror image. Particles exactly under the apex have never rolled down either slope and a discontinuity of transmission properties can be expected there. We take this history-dependence into account in the simplest way possible, by arguing that C for a volume element is “frozen in” at burial: the *Perfect Memory* assumption. Since each volume element is buried with the same value $S = 1$ this rules out any dependence of C on the magnitude of S : $C(\Upsilon, \Psi, \text{sign}(S)) = 0$.

Choice of constitutive relation: For a pile constructed from a point source, the main consequence of the perfect memory assumption is that (in two dimensions) there exists one invariant property of the stress tensor, depending on the texture as created during deposition, that remains constant after additional layers are added. Traditionally it is (often) supposed that the material remains everywhere at incipient failure (IFE), i.e. the CR is that $\Upsilon = 1$ everywhere. Numerical solution of the differential equations [4] shows that this gives a stress hump in the middle of the pile (fig. 3) rather than a dip. The same holds in $d = 3$. An alternative CR, used recently by Bouchaud, Cates

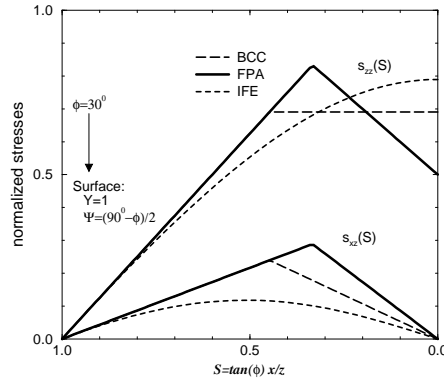


Fig. 3. Reduced shear stress and reduced normal stress as a function of scaling variable $S = \tan(\phi)x/z$ for a symmetrical two dimensional sandpile with a surface obeying IFS with an angle of repose $\phi = 30^\circ$. Results for the IFE model (found numerically), the FPA model, and the BCC model are compared.

and Claudin (BCC) [6] is to assume σ_{xx}/σ_{zz} is a constant. This closure does not break the symmetry between the left and right halves of the pile; the CR is smooth crossing the central axis. BCC gives, in two dimensions, a flat plateau in the stress.

FPA model: More successful is the FPA model. In this model we assume that it is not Υ , but the orientation of the stress tensor Ψ which remain fixed from the time of burial. This gives a dip (fig. 3). In three dimensions, for a pile with rotational symmetry, we need one additional CR to close the problem. There are some well-known candidates for this secondary closure, including equality of the radial and tangential normal stresses, or that the stress tensor is uniaxial ($\sigma_2 = \sigma_3$). Fortunately, the results do not depend much on which secondary closure is used; both choices are compared with the experimental data in fig. 1. There are no adjustable parameters; the agreement, though not perfect, is remarkably good.

3 From Principal Lines of Force to FPA and OSL

In order to motivate the FPA assumption from a more microscopic point of view, let us first step back from our *tensorial* approach to the *scalar* model due to Edwards and coworkers [5] which itself is motivated by the “stress paths” seen in experiment.[7–9] In this model the force arising from an additional small load travels along a straight line of constant angle toward the base of the pile; lines in the left and right half are mirror images. These “principal lines of force” make up a series of nested arches along which the weight is propagated. It can easily be shown[4] that this model has several drawbacks (mechanically

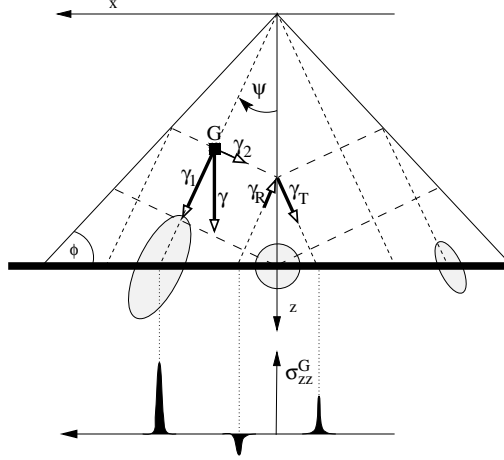


Fig. 4. Sketch of the geometry of the FPA model. The stress ellipsoid has fixed inclination angle $\Psi = (90^\circ - \phi)/2$; its ellipticity varies from zero at the centre of the pile to a maximum in the outer region. The outward and inward stress propagation characteristics are indicated by short-dashed and long-dashed lines; these are at rightangles and coincident with the principal axes of the stress ellipsoid. The forces propagating from an additional source term γ at G along the network of characteristics are shown along with the resulting vertical forces σ_{zz}^G on the bottom plate.

unstable, overprediction of dip, etc.). On the other hand it expresses intuitively the sound physical concept of local rules which tend – due to the memorized history – to propagate stresses outwards.

Similar physics arises in our FPA model. To see how this happens, we need to consider the characteristic rays for stress propagation. These can be found by combining the stress continuity equation with the FPA closure in cartesian, which reads

$$\frac{\sigma_{xx}}{\sigma_{zz}} = \eta + \mu \frac{\sigma_{xz}}{\sigma_{zz}} \quad (4)$$

with the coefficients $\eta = 1$ and $\mu = -2 \cot(2\Psi)$. (Note that μ changes sign on the symmetry axis.) The result is a wave equation in two dimensions, for which the characteristics are straight lines. In the FPA model, the directions of these characteristics are the same everywhere within each half of the pile, and *coincide with the principal axes themselves*. This is shown in fig. 4. The discontinuity in the orientation of the characteristics at the centre line leads to nontrivial reflection and transmission rules there. [4]

The physical outcome is that the stresses in the FPA model indeed propagate down a set of nested arches in the sense that there are no shear forces acting between one arch and its neighbours. However, there are normal forces, without which the outer (incomplete) “arches” would fall down. This is an appealing mechanical picture. However, it still does not explain why it is the

principal axes that get remembered when an element of sand is buried.

Fixed Characteristics (OSL) At this stage we take a step back in our set of assumptions, and generalize the FPA model slightly. There the characteristics coincide with the principal axes and hence are orthogonal. In the broader class of OSL models (from “Oriented Stress Linearity”) orthogonality is not assumed but the constraint of fixed characteristics is retained. In terms of Cartesian coordinates the OSL assumption leads to linear relationships between reduced stress components, as in eq. (4), but with free coefficients η and μ . But the repose angle can be found in terms of these [4], so the OSL model has one free parameter (say η). This fixes the anisotropy on the centre line of the pile (where $\sigma_{xz} = 0$ by symmetry) and is unity for FPA. (This allows the stress tensor to have simultaneously the principal axes pertaining to the right and left halves of the pile.) In the general OSL case, the principal axes vary smoothly through the centre-line, though the characteristics do not. Experimentally fig. 1 shows that the data is well-fit by η values close to unity. [4] A possible physical interpretation of OSL connects the characteristics with the average directions of the stress paths. In this view the *mean directions* of these stress paths remain fixed at the time of burial and is unaffected by distant loads. In the continuum limit these average directions become characteristics of the stress propagation equations, which OSL takes to be fixed. This feature is not shared by, for example, the IFE model and is not obvious. However, it is a testable hypothesis (for example by simulation [12]) whose confirmation would go a long way to confirming the modelling strategy we have used. It is important that we require only the *average* directions of the stress paths to be fixed; it seems likely that individual paths might be subject to strong noise effects [13,14].

4 Conclusion

We have described a new continuum approach to the modelling of stress propagation in static granular materials composed of hard grains in frictional contact. We reject traditional elasto-plastic concepts in favour of a local, scale-free constitutive relation among stresses which “memorizes” invariant properties of the stress tensor. These we associate with the local packing of grains: *the constitutive equation encodes the construction history*. For a pile constructed from a stationary point source, we favour the FPA model where the eigendirections of the stress tensor ellipsoid are fixed at the moment of burial. This surprisingly simple model gives good quantitative agreement with published experimental results. There is no adjustable parameter in this model. Although we have no deep justification for the FPA assumption, it is one of a broader class of models (OSL) for which the characteristics of stress propagation are fixed at burial. This suggests an appealing link with the concept of microscopic stress

paths; the OSL model would suppose that the mean orientation of such paths is invariant under loading.[14]

Acknowledgement

The authors are indebted to their collaborators J.-P. Bouchaud and P. Claudin who were closely involved in the work reviewed above. They also thank S.F. Edwards, R.M. Nedderman and J.M. Rotter for illuminating discussions. This work was funded in part by EPSRC under Grant GR/K56223 and in part by the Colloid Technology programme.

References

- [1] J. Smid and J. Novosad, in *Proc. of 1981 Powtech Conference, Ind. Chem. Eng. Symp.* **63**, D3/V/1-D3/V/12 (1981).
- [2] T. Jokati and R. Moriyama, *J. Soc. Powder Technol. Jpn.*, **60**, 184 (1979); R. Brockbank, J.M. Huntley and R.C. Ball, in preparation; F. Cantelaube, J. Goddard, *Proc. of Powders and Grains*, Durham/USA, R.P. Behringer, J.T. Jenkins (eds.), Balkema (1997).
- [3] J.P. Wittmer, M.E. Cates, P. Claudin, J.-P. Bouchaud, *Nature*, **382** (1996) 336.
- [4] J.P. Wittmer, M.E. Cates, P. Claudin *J. Physique I*, **7** (1997) 39-80.
- [5] S. F. Edwards and R. B. Oakeshott, *Physica D*, **38**, 88 (1989); S. F. Edwards, C. C. Mounfield, *Physica A*, **226**, 1,12,25 (1996).
- [6] J.-P. Bouchaud, M. E. Cates, P. Claudin, *J. Physique II*, **5**, 639 (1995).
- [7] P. Dantu, *Ann. Ponts. Chausses*, **4** (1967) 144.
- [8] C.H. Liu, S.R. Nagel, D.A. Schecter, S.N. Coppersmith, S. Majumdar, O. Narayan, T.A. Witten, *Science*, **269**, 513 (1995).
- [9] S.N. Coppersmith, C.H. Liu, S. Majumdar, O. Narayan, T. A. Witten, *Physical Review E*, **53**, No.5 (1996) 4673.
- [10] R. M. Nedderman, *Statics and Kinematics of Granular Materials*, Cambridge University Press (1992).
- [11] C. A. Coulomb, *Mem. de Math. de l'Acad. Royale des Sciences* **7**, 343 (1776).
- [12] V. Buchholtz, J.P. Wittmer and M. E. Cates, in preparation.
- [13] P. Claudin and J.-P. Bouchaud, *Phys. Rev. Lett.* **78**, 231 (1997).
- [14] P. Claudin, J. P. Bouchaud, J. P. Wittmer, M. E. Cates, in preparation.